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Coherent Quantum Logic

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The von Neumann quantum logic lacks two basic symmetries of classical logic, that between sets and classes, and that between lower and higher order predicates. Similarly, the structural parallel between the set algebra and linear algebra of Grassmann and Peano was left incomplete by them in two respects. In this work a linear algebra is constructed that completes this correspondence and is interpreted as a new quantum logic that restores these invariances, and as a quantum set theory. It applies to experiments with coherent quantum phase relations between the quantum and the apparatus. The quantum set theory is applied to model a Lorentz-invariant quantum time-space complex.

1. A CLASH OF LOGICS

The way we express logical ideas in ordinary quantum theory and in the von Neumann quantum logic conflicts with the theory of Grassmann and Peano, beautifully expounded and developed by Barnabei et al. (1985).

Suppose that ψ and ϕ are creators (that is, creation operators, kets, or psi vectors) for a quantum of odd statistics, say an electron. For simplicity, assume ψ and ϕ to be orthogonal.

Then, according to von Neumann and Dirac, the incoherent superposition "ψ or ϕ," the disjunction of ψ and ϕ, which may also be called their inclusive-or combination, is represented by the projection operator or *projector* $P=\psi\psi^*+\phi\phi^*$ upon the two-dimensional subspace $R=\text{span}(\psi, \phi)$ spanned by ψ and ϕ. We may regard $\psi\psi^*$, $\phi\phi^*$, and $\psi\psi^*+\phi\phi^*$ as representing classes of electrons. In the quantum logic of von Neumann, the fundamental entities are projectors like $P=\psi\psi^*$ and $Q=\phi\phi^*$ and the basic operation is incoherent superposition $P\cup Q$.

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Now, Grassmann and Peano (1888) insist that the proper way to represent the linear space R is not by the incoherent superposition $P \cup Q$, but by a product, the Grassmann product of ψ and ϕ, which I write as $\psi \vee \phi$, as do Barnabei et al. (1985), exactly because of this disjunctive interpretation.

The Grassmann product $\psi \vee \phi$ may be expressed by the antisymmetrized tensor product:

$$\psi \vee \phi = \Sigma_{-} \psi \otimes \phi$$

where Σ_{-} is the antisymmetrizer. But this product is already used for another purpose in quantum theory. It represents maximal information about a pair of electrons, not submaximal information about one electron. In the usual quantum theory $\psi \vee \phi$ represents a set of electrons, not a class. It is essential here that electrons have odd statistics.

There is thus the following rather strange difference between classical logic and the von Neumann quantum logic. In classical logic, one algebra, the lattice of subsets of the sample space, may express either submaximal information about a single entity or maximal information about sets of entities. In the von Neumann quantum logic two quite different algebras, one a lattice and the other a Grassmann algebra, are used for these respective purposes. Briefly, the classical algebras of (finite) sets and classes are isomorphic, the usual quantum ones are not. A basic symmetry of the classical theory is broken in the quantum theory.

This break in symmetry is not a necessary consequence of the difference between the two logics, the quantum principle of superposition. It is a historical accident, and I repair it here by revising the theory of classes. I call the new quantum logic *coherent*, and the old one incoherent, after their basic superposition processes.

Curiously, there is a similar discord in the writings of Grassmann and Peano themselves. They develop set theory and linear geometry in parallel (Table I), a parallel that first becomes a correspondence in function as well

Table 1. Correspondence Between Set Algebra and Linear Algebra of Peano and Grassmann[a]

	Set algebra	Linear algebra	
Join	$\cup$	$\vee$	Disjoin
Meet	$\cap$	$\wedge$	Conjoin
Null set	$\varnothing$	$\mathbb{C}$	1-dimensional space
Brace	$\{\cdots\}$	?	?
—	None	$+$	Sum

[a] Peano writes ιX for $\{X\}$ and calls $\vee$ and $\wedge$ the progressive and regressive products, respectively.

as form in quantum logic. But they do not perfect this correspondence. Peano builds his set algebra on the join operation $A \cup B$ and his linear algebra on the Grassmann product $\alpha \vee \beta$. But the Grassmann product is nilpotent, $\alpha \vee \alpha = 0$, while the join is idempotent, $A \cup A = A$. The two therefore do not correspond. The true set-theoretic correspondent of the Grassmann product $\alpha \vee \beta$ of spaces α and β is the *disjoint union* of sets A and B, which I therefore write as $A \vee B$. I call $A \vee B$ the *disjoin* of A and B for short. They would have been more consistent had they found classical set theory on $\vee$ than $\cup$, and I do this here. This does not make set theory stronger. Indeed, it makes it more complicated to express relations of inclusion or implication within the theory. I shall not address this problem here.

There is a second important discord between classical logic and von Neumann's quantum logic, and this, too, has its roots in a disharmony between the set and linear algebras of Peano. Peano's set theory has a basic generative process, designated by Cantor with the brace $\{\cdots\}$ and by Peano with the operator ι, which forms from each set A a new set $\{A\} = \iota A$ whose sole element is A. Without the brace we can make nothing from the null set using $\vee$ alone; with it, we can make all sets (by transfinite induction for infinite sets).

Yet neither Grassmann nor Peano nor von Neumann provides a similar generator for linear spaces. From the correspondent of the null set, the linear space $\mathbb{C}$, one can make nothing new using the $\vee$ product alone. Grassmann's linear algebra is much weaker than his set algebra.

Accordingly, after formulating a coherent quantum logic, I introduce into it a quantum operator corresponding to the $\{\cdots\}$ of Cantor and the ι of Peano. This perfects the correspondence between sets and spaces of Grassmann, Peano, and Barnabei et al. (1985), and strengthens the double algebra of Grassmann and Peano. I designate the new algebra by SET because in quantum theory, it may be interpreted as a new and stronger quantum logic, a quantum set theory.

SET raises questions. If the quantum disjunction is expressed by $\boldsymbol{\psi} \vee \boldsymbol{\phi}$, how shall an electron pair be described? What use is the brace operator? What new physical concepts are suggested by this extension of linear algebra? I answer some of these here.

Peano explicitly proposed his set theory as a universal language for mathematics. Possibly SET is a universal language for quantum theory.

Below I first recall the algebras of the usual quantum theory and of the quantum logic of von Neumann (Section 2) and classical set theory (Section 3) and compare them (Section 4). This reveals deficits in the quantum language we presently use. I repair them with the new quantum language, based on the ideas of Grassmann and Peano rather than von

Neumann, in Section 5. I translate some concepts of the usual quantum kinematics into this new quantum language in Section 6, leaving much to be done in this direction. Most of the predicates of the new quantum language have no correspondent in the old. The new predicates (Section 7) are perceived by new kinds of experiments. The results and possibilities are discussed in Section 8.

2. INCOHERENT QUANTUM LOGIC

I call the object of experimental study the *endosystem*, and the experimenter and the apparatus the *exosystem*. This division of the entire system is called the quantum partition. In the quantum theory of Dirac, each system X is associated with a Hilbert space $H(X)$. Each vector ψ in $H(X)$ represents a coherent quantum creation process. We reason about the system with $H(X)$ in ordinary quantum physics, as we reason about it with a phase space in classical physics, and this constitutes a species of logic, although its basic operations are those of a linear space rather than a lattice. It is customary to describe every pure (singlet, nondegenerate) quantum class (or predicate) of a system X by a projection or *projector* p onto a ray in a Hilbert space $H = H(X)$ associated with X. If ψ is a unit vector and ψ^* is the dual vector in the space DH, the dual space to H, then

$$p = \psi\psi^*$$

is such a projector. The probability of a transition $p \to p'$ is

$$\text{Prob}(p \to p') = \text{Tr}(pp'')$$

A general class is represented by a projector P onto a higher dimensional subspace, which may be expressed as sums of pure projectors,

$$P = p + p' + \cdots$$

which are (pairwise) orthogonal:

$$pp' = pp'' = \cdots = 0$$

The probability of a transition $P \to P'$ is given by

$$\text{Prob}(P \to P') = \text{Tr}(PP')/\text{Tr}(P)$$

Evolution processes of an isolated quantum are represented by unitary operators in the same space H. More general transformations are represented by more general linear operators in the algebra $\mathbf{L}H(X)$ of linear operators on $H(X)$.

If the quantum X is itself an ensemble of subquanta x of odd statistics, then

$$H = \vee h$$

The Grassmann algebra over the Hilbert space h of x. If ψ and ψ' are unit vectors in h, then the Grassmann product

$$\Psi = \psi \vee \psi'$$

is associated with a projector

$$P = \Psi\Psi^*$$

representing a pure class of pairs. $\vee H$ is graded thus: 1 has grade 0. The vectors ψ of H are of grade 1. The $\vee$ product of g vectors is of grade g.

One Hilbert space algebra $\mathbf{L}H(X)$ thus represents such diverse entities as classes, dynamical variables, and transformations associated with the quantum X. We may represent the quantum itself in $\mathbf{L}H(X)$ by the unit operator $\mathbf{1}$. Sets of X are represented, however, in $\mathbf{V}H(X)$ and $\mathbf{LV}H(X)$.

The von Neumann quantum logic abstracts from the Hilbert space algebra $\mathbf{L}H(X)$ the projectors $P, Q, \ldots$, the operation of disjunction or union $P \cup Q$, and the operation of complementation or negation $\sim$. Its fundamental superposition operation $\cup$ represents incoherent superposition. I therefore call this theory incoherent quantum logic.

3. SET THEORY

Set theory is used by many as a universal language for mathematics. I build sets from their elements not with the operation of union $A \cup B$, as is customary, but with the *disjoin*, defined only when A and B are disjoint and then equal to the join or union $A \cup B$. This makes no significant difference in the classical theory.

I define

$$A = 0 \quad \text{to mean} \quad A \text{ is undefined}$$

$$A = 1 \quad \text{to mean} \quad A \text{ is the null set}$$

(because $\vee$ is our product and the null set is its identity). Then the disjoin may be expressed in familiar terms by

$$\begin{aligned} A \vee B &= A \cup B \qquad \text{if } A \cap B = 1 \\ &= 0 \qquad\qquad \text{otherwise} \end{aligned}$$

Its characteristic properties are associativity, commutativity, and nilpotency:

$$A \vee A = 0$$

I designate the power set of a set X by $\vee X$ since it consists of all $\vee$ products of elements of X.

The other set operation needed to generate all sets is the brace $\{A\}$. Here $\{A\}$ is the set whose only element is A. Sometimes it is convenient to write Cantor's brace $\{A\}$ as Peano's ιA. (In earlier papers I use B instead of ι.)

3.1. Inductive Construction

For simplicity (and also as a matter of principle) I consider only the class of *finitary* sets here, that is, those sets generated from the null set by a finite number of operations of disjoining and bracing. I call this close SET. Henceforth unless otherwise mentioned all sets belong to SET. SET is generated from 1 according to the following postulates:

P1. 1 is in SET.
P2. If A and B are in SET, then $\{A\}$ and $A \vee B$ are.
P3. $\{\cdots\}$: SET $\to$ SET is injective with $\{A\} \vee \{A\} = 0$ for every $A \in$ SET. SET (with multiplication $\vee$) is a commutative semigroup with 0 and 1.
P4. There are no relations among elements of SET except those that follow from P1–P3.

In particular,

$$0 \vee A = 0$$

$$1 \vee A = A$$

$$A \vee A = 0$$

SET is graded thus. 1 has grade 0. Every brace $\{A\}$ has grade 1. The $\vee$ product of g sets of grade 1 has grade g. The grade of A is written $\mathbf{G}A$. This grade is also called cardinality and multiplicity, depending on interpretation. The grade-g sets that belong to any class P form a subclass of P designated by $[\mathbf{G} = g]P$.

A set is a pure description of a possibly plural entity. Each set symbol may also be used as a class, a possibly impure description of a single entity. For example, if A and B are pure first-grade sets, then $A \vee B$ may be used either as a pair $\{A, B\}$ or an impure description of a singleton as being either A or B. In the set interpretation the grade is called cardinality; in the set interpretation, multiplicity or statistical weight.

When X and Y are interpreted as classes, the probability of a transition $X \to Y$ is the ratio of statistical weights

$$\text{Prob}(X \to Y) = \mathbf{G}(X \cap Y)/\mathbf{G}X$$

For pure (grade 1) classes p, p' the transition probability $\mathrm{Prob}(p \to p')$ is 0 unless $p = p'$ and is then 1.

In set theory every entity is a set, and may be found in the universal sample space SET. The natural numbers $\mathbf{1}, \mathbf{2}, \ldots \in \mathbb{N}$ are usually represented as sets by taking $\mathbf{1} = \{1\}$ and defining (inductively) the successor $\mathbf{N}'$ of $\mathbf{N}$ as $\mathbf{N}' := \mathbf{N} \vee \{\mathbf{N}\}$. This then fixes the description of all entities that may be expressed in terms of numbers.

In set theory, all entities are described in the one class SET. The invariance group of SET is trivial. Like the integers, sets are intrinsically different from each other.

3.2. Conjoin

Relative to any set U serving as universe of discourse, there is a dual operation to the disjoin $\vee$ called the conjoin $\wedge$. The set U defines an involution mapping each set $A \subset U$ into its complement A' relative to U, mapping 1 into $1' = U$, and each point p of U into a *copoint* p'. By

$$A \wedge B = C$$

we mean that

$$A' \vee B' = C'$$

Schematically, we may write

$$\wedge = \vee'$$

4. THE LOST SYMMETRIES OF QUANTUM LOGIC

Let us now compare the quantum and set algebras. It is convenient to do so in terms of symmetries or invariance principles.

4.1. Hierarchic Invariance

In set theory, for any entities X, $Y, \ldots$ there is a new entity, the set $\{X\} \vee \{Y\} \vee \cdots$, usually written $\{X, Y, \ldots\}$, having them for elements. For its construction it suffices, if we regard the $\vee$ product as a given, to define the brace operation $\{\cdots\}$, a map from SET to $[\mathbf{G} = 1]$. This map is an injection of $[\mathbf{G} = 1] \to [\mathbf{G} = 1]$; that is, it is 1-1 into.

If a set is interpreted as a class of predicate p, then $\{p\}$ may be interpreted as a predicate about predicates, or second-order predicate, true by definition for the predicate p itself and for no other.

By a hierarchic invariance I mean an endomorphism of the predicate algebra that increases order (which is the rank of the corresponding set) and thus goes up in the hierarchy of sets. In classical logic the brace operation $\{\cdots\}$ on pure predicates may be uniquely extended to a hierarchic invariance.

The von Neumann quantum logic lacks a brace operation, and treats quantum predicates, which are first order, differently from predicates about quantum predicates, which are second order. There is a superposition principle for predicates about quanta, but not for predicates about predicates, or higher order predicates. The first-order logic is quantum, but the second-order one is classical. The logic and metalogic of the quantum theory are different. Sometimes this has been considered sufficient reason to disqualify quantum logic as a logic at all. In any case, the present quantum theory lacks hierarchic invariance, and the lack has been felt.

4.2. Extensional Invariance

In set theory, every set S is the *extension* of some predicate P, that is, the collection of all entities enjoying the property P. I write $S = \text{Ext}\ P$ for this relation. Corresponding to the disjoin $A \vee B$ of sets there is a disjoin $\alpha \vee \beta$ for predicates, defined only for mutually exclusive predicates, and then equal to their ordinary disjunction. Then

$$\text{Ext}(\alpha \vee \beta) = \text{Ext}(\alpha) \vee (\text{Ext}\ \beta)$$

Thus, Ext is an injection of the class v-algebra into the set v-algebra. I call this familiar and classically trivial invariance property *extensional invariance* of the logic.

In von Neumann's quantum theory, a predicate about the quantum X is represented by a subspace of $H(X)$, and a set of X's (or more precisely, a pure predicate about a set of X's) is represented by rays in $\mathbf{V}X$. The extension of a predicate represented by a subspace P of $H(X)$ is represented by the Grassmann element of $\mathbf{V}H(X)$ representing the subspace P; that is, the Grassmann product of the vectors in a basis for P. More uniquely, it is represented by the ray in $\mathbf{V}H(X)$ containing that Grassmann element. There is no predicate about X corresponding to the general ray in $\mathbf{V}H(X)$, only to the rays consisting of products of vectors, the Grassmann elements that Grassmann called "real." The von Neumann quantum logic does not possess extensional invariance. The peculiar absence of any isomorphism between the theories of sets and classes in quantum theory has caused concern.

4.3. Projective Invariance

The classical predicate algebra of entitity X is invariant under the group of 1–1 maps of predicates into predicates respecting the $\vee$ product. This is just the group of permutations of the points of the sample space SX. The negation operation of the classical logic is invariant under the group of $\vee$ (the group of all maps that preserve the disjoin relation).

The usual quantum predicate algebra is not invariant under the group of v. In quantum theory this group is the projective group of the linear space $H(X)$ and does not respect the orthogonality relation of $H(X)$, nor therefore the orthocomplement relation of $H(X)$, which is the negation of the quantum logic.

This somewhat recondite invariance property merely rationalizes my choice below of the unimodular group over the unitary group. My ultimate reason for this preference is that relativistic binary spinors have a unimodular group, and I wish to accommodate a quantum theory to them.

5. QUANTUM SET THEORY

As an example of a coherent quantum logic, I build a quantum set theory that strictly parallels the classical one, taking as basic two linear operations of disjoin v and brace$\{\cdots\}$ corresponding to the two of classical set theory. For simplicity and familiarity I work with complex coefficients. Then in addition to v and $\{\cdots\}$ as operations on creators there are the complex numbers $\mathbb{C}$ acting as multipliers and the operation of vector addition +. These are the basic operations of the quantum set theory.

5.1. Inductive Construction

The most general creator of quantum set theory—called a plexor for its subsequent topological interpretation as a simplex or complex—is made from the complex number 1 by the operations of v, $\boldsymbol{\{\psi\}} := \boldsymbol{\iota\psi}$, $\mathbb{C}$, and +, according to the following postulates **P1–P4**:

P1. 1 is in **SET**.

P2. If A and B are in **SET**, then $\{A\}$ and $A \mathrm{v} B$ are.

P3. $\boldsymbol{\iota}$: **SET** → **SET** is a linear operator. **SET** (with $\mathbb{C}$, v, +) is a complex Grassmann algebra over **{SET}** (the $\boldsymbol{\iota}$ image of **SET**).

P4. Elements of **SET** obey no relations but **P1–P3** and their consequences.

In particular,

$$0 \mathrm{v} A = 0$$

$$1 \mathrm{v} A = A$$

$$\{A\} \mathrm{v} \{A\} = 0$$

for any plexor A in **SET**.

P1–P3 and the above relations correspond well to P1–P3 and the relations following them. It would be easy to frame the two systems so that they differ only in the principle of superposition.

There then remains an important discord. SET has no symmetries, while **SET** has complex conjugation symmetry C.

Since an exact symmetry implies unobservable quantities, an operational theory should have none. This C symmetry is a consequence solely of our choice of field. I conjecture that a more fundamental coefficient system is real, and is in fact the integers. With that choice, SET would be not a linear space, but merely a free Abelian group. I drop this possibility for now.

I designate the Grassmann grade in SET by **G**.

The quantum set described by SET may be called abstract. If there is any quantum entity X that is not a set, there is also a concrete set theory SET(X) founded on X, containing X and also sets of X's, for example. I do not require this concept here.

Just as grade **G** counts factors, there is a rank **R** that counts nested braces. **R** is a linear operator on SET, defined inductively:

$$\mathbf{R}1 = 0$$

$$\mathbf{R}\boldsymbol{\iota} = \boldsymbol{\iota}(\mathbf{R}+1)$$

$$\mathbf{R}(\boldsymbol{\psi}_1 \vee \boldsymbol{\psi}_2) = \sup(R_1, R_2)(\boldsymbol{\psi}_1 \vee \boldsymbol{\psi}_2) \qquad \text{if} \quad \mathbf{R}\boldsymbol{\psi}_i = R_i \boldsymbol{\psi}$$

For example, we may construct a family $\mathbb{N}^Q$ of quantum sets $1^Q, 2^Q, \ldots$ corresponding to the natural-number sets $\mathbf{1}, \mathbf{2}, \ldots \in \mathbb{N}$ of Section 3.1 by taking $1^Q = \{\mathbf{1}\}$ and defining (inductively) the successor $\mathbf{N}'$ of $\mathbf{N}$ as $\mathbf{N}' := \mathbf{N} \vee \{\mathbf{N}\}$, for any $\mathbf{N}$ in $\mathbb{N}^Q$. In just the same way, omitted as obvious, we make quantum sets corresponding to the integers $\mathbb{Z}$, designating the quantum set corresponding to the integer n by n^Q, and the entire family by $\mathbb{Z}^Q$.

Conspicuously lacking is a preferred inner product of SET. In that sense this is a nonunitary quantum theory. I assume (in order to connect with the theory of Bergmann spaces of Section 5.2) that each experimenter brings in an inner product, reducing the group from the general linear group to the unitary one. We recover the usual unitary quantum theory by restricting experimenters to those with a common metric.

To express a relativistic theory in SET, we assign to each exosystem E a subspace of SET and a basis $\boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_N$ for that subspace. Permutations of this basis, unitary transformations of the subspace into itself, and general transformations of the subspace, possibly into another, are called transformation theories of the third, second, and first levels, respectively, in Finkelstein (1987).

5.2. Conjoin, Grassmann Space, Peano Algebra

To construct a conjoin operation, we must choose a finite-dimensional subspace U of grade 1 plexors in SET as universe of discourse and define the conjoin $\wedge$ relative to U. For this we must first give U the structure of a *Grassmann space.*

A Grassmann space, as defined by Barnabei et al. (1985), is a linear space U provided with an antisymmetric scalar form, the *Grassmann bracket* $[\psi_1\psi_2 \cdots \psi_N]$, of maximum degree. The degree N is then the dimension of the linear space U. The coefficients of the Grassmann bracket form an antisymmetric covariant tensor δ with N indices, necessarily a multiple of the Levi-Civita tensor density ε, by a pseudoscalar ρ: $\delta = \rho\varepsilon$. Such a Grassmann space is used as a spinor space in Misner et al. (1973), for example (with $N = 2$ and with Grassmann tensor δ designated by ε).

Let DU be the dual space to U, and $D\mathbf{V}U$ the dual space to $\mathbf{V}U$. It is clear that $D\mathbf{V}U$ is also a Grassmann algebra with product $\vee$, over DU. The Grassmann tensor δ defines a mapping $\delta: UU \to D\mathbf{V}U$, carrying plexors of grade g into dual plexors of grade $N-g$. The mapping δ is called "lowering indices with δ." The inverse map δ^{-1} is called "raising indices with δ^{-1}." The δ^{-1} image in $\mathbf{V}U$ of the scalar 1 in $D\mathbf{V}U$ is a pseudoscalar

$$I := \delta^{-1}1$$

a product of all the elements of a basis for U, and the Grassmann volume element of the space U. I designate by U' the linear subspace of $\mathbf{V}U$ consisting of the elements of $\mathbf{V}U$ of Grassmann grade $N-1$, sometimes called covectors or pseudovectors.

The conjoin $\wedge$ is the δ^{-1} transform of the disjoin $\vee$. That is, whenever covectors δA, δB, δC obey

$$\delta C = \delta A \vee \delta B$$

we set

$$C = A \wedge B$$

Schematically put,

$$\wedge = \delta^{-1}(\vee)$$

The Grassmann algebra $\mathbf{V}U$ over U with coefficients $\mathbb{C}$ and product $\vee$ is also a Grassmann algebra over U' with coefficients $\mathbb{C}I$ (the complex multiples of I) and product $\wedge$. Following Barnabei et al. (1985), I call such a doubly Grassmann algebra a *Peano* algebra.

To recapitulate, the linear space U has group $GL(N, \mathbb{C})$. Giving a metric would reduce this group to $U(N, \mathbb{C})$. I do not do this. Instead, I asssume a Grassmann form, reducing the group to $SL(N, \mathbb{C})$.

The assumption that the world has such a local structure group leads to a version of Kaluza-Klein theory in which the world is described as a *Bergmann manifold*; meaning, a real, differentiable manifold M, provided at each point $x \in M$ with a Grassmann space $\Psi(x)$ of spinors, and a differentiable map σ, called the spin vector, from the Hermitian tensors on $\Psi(x)$ to $dM(x)$, the tangent space to M at x; see Bergmann (1957), Finkelstein (1986), and Holm (1986).

5.3. Quantum Time-Space Complex

As an application of this calculus, I construct a Lorentz-invariant quantum model for time space. It resembles the Feynman and Hibbs (1965) two-dimensional checkerboard model, but is composed of triangles instead of squares, is a four-dimensional time-space instead of two-dimensional, and its Lorentz invariance will be exact instead of approximate. It resembles more closely the quantum checkerboard of Finkelstein (1967), which also has an exact local Lorentz invariance, but this model has a global one as well.

Let α_{mn} be a two-index family of independent monadic plexors, which shall be the vertices of the time-space complex, with $m, n \in \mathbb{Z}$. For example, we may take α_1 and α_2 to be two plexors independent of the quantum integers $\mathbb{Z}^Q$ and set

$$\alpha_{mn} = \{\{m^Q \vee \alpha_1\} \vee \{n^Q \vee \alpha_2\}\}$$

for all $m, n \in \mathbb{Z}$.

Then the triangles

$$\beta_{mn} := \alpha_{mn} \vee \alpha_{(m+1)n} \vee \alpha_{m(n+1)}$$

and the "inverted" triangles

$$\beta'_{mn} = \alpha_{mn} \vee \alpha_{(m-1)n} \vee \alpha_{m(n-1)}$$

form the triangular complex shown in Figure 1, familiar from the game of "Chinese checkers." I brace and multiply these triangles to form a plexor

$$\Psi = \prod_{m,n} \{\beta_{mn}\}\{\beta'_{mn}\}$$

In principle, m and n should range over finite intervals—say, $M < m$, $n < M$—and only at the end should we examine the limit $M \to \infty$. Finally, the quantum time-space complex is the "world plexor"

$$W = \Psi \vee \Psi^*$$

Just as a spinor ψ may be called a linear square root of a time-space vector $w = \psi \vee \psi^*$, W may be called a topological square root of time-space itself.

I now consider the Lorentz invariance of W. Because Ψ is made of triangles, when we fix one vertex and one triangle we define naturally a group $SL(2, \mathbb{C})$ that transforms the other two vertices of the triangle into linear combinations of themselves, called the local spin group. The complex Ψ is thus provided with a natural isomorph of $SL(2, \mathbb{C})$, the universal covering group of the Lorentz group, at each vertex of each triangle. This is the correspondent in the quantum theory of the local action of $SL(2, \mathbb{C})$ on the local spinor space of the time-space Bergmann manifold. Each element u of the group $SL(2, \mathbb{C})$ at vertex α of triangle β in Ψ acts thus.

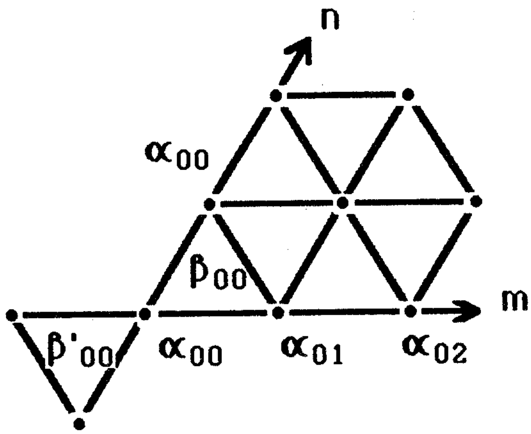

Fig. 1.

Let u be presented as a unimodular matrix u_n^m with $m, n = 0, 1$. Then at vertex $\alpha = \alpha_{00}$ in triangle $\beta = \beta_{00}$,

$$D(\beta, \alpha, u): \quad \alpha_{1m} \to {}_{1n}u^n_m,\ \alpha_{00} \to \alpha_{00}$$

That is, the two vertices of β distinct from α are acted on by u as if they were "up" and "down" unit spinors $\uparrow$ and $\downarrow$. The action for other vertices α of the same triangle is found by cyclic permutation of the three vertices. The action for other triangles β_{mn} is found by translation of the above action in the (m, n) plane. The action for "inverted" triangles β'_{mn} is found by the inversion $m \to -m$, $n \to -n$.

I now consider the Poincaré invariance of W. First I extend each local spin group $D(\beta, \alpha, u)$ to a global one $G(\beta, \alpha, u)$ acting on the linear space spanned by all the simplices of Ψ, expressing Lorentz transformations not merely of one triangle, but of the entire time-space complex about the selected origin, a subgroup of the Poincaré group. It suffices to construct the global transformation for the vertex α_{00} of the triangle β_{00} as origin, and then transfer the work to other vertices and triangles by cyclic permutation, translation, and inversion as above.

I extend the action of the spin group on the vertices α_{mn} from its action on the three vertices α_{00}, α_{01}, α_{11} of β_{00} so as to respect the topology of Ψ. It suffices to consider the six infinitesimal generators σ_{st} of the spin group. Of these, the three boosts σ_{0s} are imaginary multiples of the three rotations σ_{tu} for $s, t = 1, 2, 3$, so that by linearity it suffices to consider the rotations. I consider first the generator $\sigma_{23} = \sigma_1$, which simply interchanges α_{01} and α_{10}, leaving α_{00} fixed. Its global extension must therefore interchange the triangles β_{01} and β_{10}. Since these have one point α_{11} in common, this must be fixed, and α_{12} must be interchanged with α_{21}. In this way we propagate the action of σ_1 to the entire (m, n) plane. It is a reflection in the straight line through $(0, 0)$ and $(1, 1)$.

Similarly, $i\sigma_2$, which interchanges α_{01} and α_{10} with change in sign, must be a reflection with change in sign in the same line. And finally the actions of σ_1 and σ_2 define that of $\sigma_3 = -i\sigma_1\sigma_2$. This completes the definition of the global action $G(\alpha, \beta, u)$ of $SL(2, \mathbb{C})$.

Every Poincaré transformation is a product of Lorentz transformations about various origins. Therefore I define the quantum Poincaré group P^Q as the group generated by all the global transformations $G(\alpha, \beta, u)$ for all vertices α, triangles β, and spin transformations u.

It is clear that analogues of this construction may be carried out in every dimension N. I cannot help speculating on the physical consequences of such possibilities. A continuum approximation to such an N-dimensional plexor is presumably a Bergmann manifold, with structural group $SL(N, \mathbb{C})$ instead of $SL(2, \mathbb{C})$, and therefore with time-space dimension $n = N^2$ instead of 4. In principle, Bergmann manifolds, like Riemannian ones, may have arbitrarily high dimension. Riemannian manifolds, however, admit well-behaved dynamical principles in every dimension, and indeed with increasing number of arbitrary parameters as the dimension grows [Lovelock, 1971; Zumino, 1986; Deruelle and Madore 1986). On the contrary, it appears that Bergmann manifolds admit invariant second-order dynamical principles only for $N = 2$, $n = 4$ (Finkelstein, 1987). Then a higher dimensional world plexor cannot have a well-behaved continuum limit. It remains an attractive possibility that there are internal dimensions of microscopic extent, and that this unstable behavior of higher dimensional Bergmann manifolds is the continuum limit of a quantum dynamical process reducing the spinor dimension to $N = 2$ at the plexic level and the time-space dimension to $n = 4$, a higher dimensional analogue of a fluid filament decomposing into droplets in an atomizer.

6. COHERENT QUANTUM LOGIC

Let X be a fermionic quantum under study. Pure production processes ("creations") of X are represented by rays in $H(X)$. With the usual ambiguity we represent creations by individual nonzero vectors of $H(X)$, creators. Pure counting processes (destructions) are represented dually by coplexors, elements of the dual space $DH(\mathbf{X})$, destructors.

6.1. Predicates

Both creations and destructions correspond to pure or point predicates of classical theory, where the processual aspect is usually ignored. They may be termed initial and final pure predicates, respectively.

The Grassmann product $\psi \vee \phi$ of creators ψ and ϕ represents the disjoin of ψ and ϕ regarded as (initial) predicates. It gives the submaximal information that is conveyed in the usual quantum theory by the projector on the subspace span(ψ, ϕ) spanned by ψ and ϕ. Its entropy (in natural units) is ln 2.

The most general predicate of the usual quantum theory, a projector on some subspace P in the von Neumann theory, is now represented by a *product* plexor, the Grassmann product of the vectors in a basis for P. I return below to the interpretation of superpositions of product plexors.

The logical disjoin is expressed by the v product, and the logical conjoin by the $\wedge$ product. A formal implication relation $\alpha \to \beta$ for quantum predicates (and inclusion for classes) is expressed by the factorization

$$\beta = \alpha \vee \gamma \qquad \text{for some } \gamma$$

A negation operation requires a Hilbert structure on U. A material implication $\alpha \subset \beta$ seems to require still more structure.

6.2. Assemblies

The pair creator associated with ψ and ϕ is now $\psi \vee \phi$. This has grade 2, hence entropy ln 2, over $H(X)$, but grade 1, hence entropy 0, over $\mathbf{V}H(X)$. The most general pure creator of a set of X is an element of $\mathbf{V}H(X)$.

6.3. Variables

In the usual theory a quantum variable V is represented by a linear operator V given by a spectral sum of the values V', V'', ... in the spectrum of V and the corresponding spectral projectors $[V = V']$, ... according to

$$V = V'[V = V'] + V''[V = V''] + \cdots$$

The projector $[V = V']$ represents the predicate $V = V'$. The spectral sum encodes the disjunction,

Either the reading is V' and $V = V'$,
or
the reading is V'' and $V = V''$,
or ...

This representation is rejected by Grassmann. Each g-dimensional projector $[V = V']$ should now be replaced by a product plexor of grade g, the Grassmann product of g independent eigenvectors of the projector. I shall now assume these belong to Set and write $\{V = V'\}$ for the brace of their product.

If, however, we merely insert the plexors $\{V = V'\}$ into the spectral sum in place of the projectors $[V = V']$, we will not be able to recover the plexors from the sum alone, not even up to multipliers. We require a new reversible coding.

There is a straightforward translation of the above disjunction into quantum set theory, if we understand "the reading is V'" to be a statement about the value of some meter variable M, such as a needle position. It is

$$\{\{M = V'\}\text{v}\{V = V'\}\}\text{v}\{\{M = V''\}\text{v}\{V = V''\}\}\text{v}\cdots$$

Here a v within braces represents composition of a predicate about the meter and a predicate about the endosystem.

The old spectral sum does not mention the instrument explicitly and is useful as a generator of unitary infinitesimal transformations. The new Grassmann product is explicit and, since it uses no metric, more invariant.

7. COHERENT SOURCES

There are many more predicates in coherent quantum logic than in von Neumann's quantum theory over the same linear space. The predicates of von Neumann correspond only to the (rays of the) product vectors of the Grassmann algebra, the elements that Grassmann calls "real." Almost no Grassmann elements are factorizable into first-grade elements. I turn now to the meaning of these nonproducts.

When a plexor is interpreted as a quantum set rather than a predicate, the "unreal" elements appear as quantum superpositions of the real. For example, $\{1\}+\{1\}\{\{1\}\}$ is a quantum superposition of the monad $\{1\}$ and the dyad $\{1\}\{\{1\}\}$, with equal amplitude. (By an n-ad I mean a set with n elements, or a plexor of Grassmann grade n.) I take the concept of quantum superposition for granted here, even when different occupation numbers are superposed. This is not the problem.

When the same plexor is interpreted as a predicate, however, we find a superposition of one predicate associated with a rat and another associated with a two-dimensional subspace. The two-dimensional element is already an incoherent superposition itself. We are not used to taking coherent superpositions of incoherent ones.

I interpret these new predicates by using the new extensional symmetry between predicates and sets to convert them to sets, then interpreting the sets, and then inverting the extensional symmetry to return to predicates. For this the following concepts are useful.

7.1. Holistic and Ensemble Interpretations

A creator as predicate describes a mode of creation. There are two standard ways of relating a creator to experiment to keep in mind, Bohr's and von Neumann's, the individual and the ensemble.

They are much like two ways in which one relates a classical distribution function in phase space to experiment [as in Schrödinger (1948)]. First a distribution function represents a population consisting of a number of individual endosystems, each in its own heat bath. But heat baths may be quite varied and complex, and difficult to analyze. Therefore we reinterpret the distribution function to describe a "warehouse" or ensemble of similar endosystems, so that each is in a standard heat bath consisting of replicas of itself.

I shall refer to these two as the holistic and the ensemble interpretations of the distribution function, respectively. Evidently the holistic interpretation is more general than the ensemble one. A population of similar endosystems is a special heat bath, while the holistic interpretation allows for more general heat baths.

In quantum theory the environment of the quantum, including the experimenter and the apparatus, replaces the heat bath of statistical thermodynamics. Heisenberg and Bohr represent an individual process with a creator ψ, describing the experimenter, the apparatus, and the quantum, the "entire situation." We may imagine writing the ψ on a black box that emits the individual quantum, as representative of the process. Bohr's is a holistic interpretation of the quantum theory.

Von Neumann, however, replaces the process of creation, which (like a heat bath) may be quite complex and varied and difficult to analyze, by a warehouse (von Neumann's metaphor) of similar quanta. Now the mode of creation is simple and standard: random selection from a population of quanta of similar constitution. Therefore the creator may now be assigned to the variable element of the process, the population. For von Neumann, therefore, a creator representing a predicate about X is associated with an ensemble of X's. This is the ensemble interpretation of the quantum theory. Again, the holistic interpretation is more general than the ensemble one. Selection from an ensemble is only one possible creation process.

The Copenhagen interpretation permits us to attach a creator to an experiment in which a single photon comes from a given polarizer, even if the polarizer is smashed before any other photons get through. A holistic interpretation is thus also an individual interpretation. In the von Neumann approach a creator applies only to a large ensemble of identically created photons.

To avoid confusion I mention the widely used formulation in which ψ is thought (much as in Schrödinger's earliest theory) to be the endosystem itself, "collapsing" or "reducing" to another ψ in an unpredictable way when the experiment reaches the destruction phase. These concepts make the correspondence of quantum physics with classical physics quite unperspicuous and are not used by Bohr or Heisenberg. Since this formulation is often called Copenhagen but is not, I call it the pseudo-Copenhagen formulation.

7.2. Interpretation of the New Predicates

The ensemble viewpoint, however, does make it obvious how to interpret an "unreal" or nonproduct creator Ψ as a predicate. First we interpret Ψ as a set; this is easy. Then one creation process represented by Ψ consists of selecting one quantum at random from that set.

In the von Neumann quantum theory, the populations are required to be uncorrelated, of the kind that may be represented in quantum set theory by product plexors. The quanta of two consecutive drawings have no coherent phase relations.

In coherent quantum logic, however, the populations are allowed to be correlated. In case of a product Ψ, there is no such correlation and the theory reduces to the von Neumann quantum theory. The correlated cases are the new "unreal" predicates.

When in fact the individual viewpoint prevails, as when the predicate is represented by a more general process than selection from a population, any correlations between succeeding quanta from the same source must arise from correlations between each quantum and its source. Experiments with such coherent phase relations between endosystem and exosystem we may call "coherent." In the von Neumann theory, the experiment is incoherent, and has no phase relations with the emitted quantum. The prototype incoherent experiment with a proton spin is a polarizer-analyzer sequence. The creation phase of the experiment is a proton gun followed by a Stern-Gerlach magnet, and the destruction phase consists of a second Stern-Gerlach magnet followed by a proton counter.

Coherent quantum logic allows from the start for the possibility of coherent sources. The prototype coherent experiment is a fission-fusion sequence. The creation phase uses a source of (say) ground-state nuclei, followed by a surface that transmits one nucleon and reflects the others. The possibly transmitted nucleon is the endosystem. The residual nucleus remains entirely in the exosystem and coherently propagates to the destruction phase of the experiment, where it recombines with the emitted proton to form a nucleus, which then passes through an analyzer. The coherent logic reduces to the usual one when the nucleus is a single nucleon and the experiment ignores the phase between the nucleon and its source.

8. CONCLUSIONS AND BEGINNINGS

The usual Hilbert-space quantum theory of Heisenberg and Dirac, supplemented by the disjoin and conjoin Grassmann-product operations, becomes a unitary coherent quantum logic, a more general logic than the lattice theory of von Neumann. We relativize its Hilbert space metric to arrive at the unimodular coherent quantum logic of this study.

If X is any quantum and $L = L(X)$ its linear space of creators, a predicate about X in the coherent sense proposed here is represented by a ray in the Grassmann algebra $\mathbf{V}L$. The usual projection operators onto subspaces of $L(X)$ represent only the product elements of $\mathbf{V}L$ and omit their coherent superpositions.

The correspondence between set and linear algebra (or between classical and quantum set theory) is now that of Table II. The remaining difference is the quantum principle of superposition (the + operation), which naturally has no correspondent in classical set theory.

The quantum set theory presented here possesses invariance under the unimodular group, instead of the unitary group of the von Neumann theory, or the pseudo-orthogonal group of the Clifford-algebraic theory of Finkelstein and Rodriguez (1986).

Von Neumann's additive representation of submaximal information and Grassmann's multiplicative one are not mutually exclusive, but can cohabit the same theory. In the absence of a metric, both a composite quantum and submaximal information about one quantum may be represented by a product creator $\boldsymbol{\Psi}$. In the presence of a metric, a "one-body" projection operator of the usual kind can be made from $\boldsymbol{\Psi}\boldsymbol{\Psi}^*$ by tracing over all quanta but one. If $\boldsymbol{\Psi}$ is not a product but a superposition of products, the same procedure leads to a statistical operator.

We have now explored three main roads to quantum logic.

The classical inclusive disjunction (U) leads to the lattice logic of von Neumann. This logic is highly asymmetric compared to classical set theory.

The classical exclusive disjunction (xor) leads to a Clifford algebraic logic. This is less asymmetric, but requires an arbitrary choice of signature and is therefore nonunique compared to classical set theory. Moreover, its fundamental plexors transform with spin 1 (under the orthogonal group). In such a theory spin 1/2 is not fundamental.

Table II. Correspondence Between New Set Algebra and New Linear Algebra

	Set algebra	Linear algebra
Disjoin	$\vee$	$\vee$
Conjoin	$\wedge$	$\wedge$
Null	1	**1**
Brace	$\{\cdots\}$	$\{\cdots\}$
Sum	+	+

The disjoin (v) leads to the Grassmann algebraic logic presented here. This is the most plausible of the three theories: first, because of its greater symmetry; second, because the unimodular groups of the Peano algebras in SET include the time-space structural group, which is now to be regarded as the unimodular group $SL(2, \mathbb{C})$ of two-component spinors, or its higher dimensional analogues $SL(N, \mathbb{C})$; now plexors transform as spinors or hyperspinors; and third, because the isomorphism of the algebras of classes and sets within SET seems to resolve basic conceptual problems of time-space structure. From the point of view of an electron, the collection of paths in time-space is a class of alternative possibilities, while from the point of view of quantum gravity it is a set of paths actually present. In SET this one collection has one formula, as in classical logic.

The relation $v = \psi\psi^C$ between complex binary spinor ψ and real time-space four-vector v has been known since Cartan. Now it is meaningful to ask whether there is a similar relation $V = \Psi\Psi^C$ between a complex two-dimensional plexor Ψ and a four-dimensional Hermitian time-space plexor W.

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REFERENCES

Barnabei, M., Brini, A., and Rota, G.-C. (1985). On the exterior calculus of invariant theory, *Journal of Algebra*, **96**, 120–160.

Bergmann, P. G. (1957). Two-component spinors in general relativity, *Physical Review*, **107**, 624–629.

Deruelle, N., and Madore, J. (1986). Kaluza-Klein cosmology with the Lovelock Lagrangians, preprint.

Feynman, R., and Hibbs, A. R. (1965). *Quantum Mechanics Via Path Integrals*, Wiley, New York.

Finkelstein, D. (1967). Space-time code, *Physical Review*, (1969).

Finkelstein, D. (1986). Hyperspin and hyperspace, *Physical Review Letters*, **56**, 1532–1533.

Finkelstein, D. (1987). The universal quantum, Paper presented at the conference on The World View ov Contemporary Physics, Colorado State University, Fort Collins, September 25, 1986, to be published.

Finkelstein, D., and Rodriguez, E. (1986). Quantum time space and gravity, in *Quantum Concepts in Time and Space*, R. Penrose and C. Isham, eds., Oxford.

Holm, C. (1986). Christoffel formula and geodesic motion in hyperspin manifolds, *International Journal of Theoretical Physics*, in press.

Lovelock, D. (1971). The Einstein tensor and its generalizations, *Journal of Mathematical Physics*, **12**, 498.

Misner, C. W., Thorne, K., and Wheeler, J. A. (1973). *Gravitation*, Freeman, San Francisco.
Peano, G. (1888). *Calcolo geometrico secondo l'Ausdehnungslehre di H. Grassmann, proceduto dalle operazioni della logica deduttiva*, Bocca, Turin [Partial translation in H. C. Kennedy, *Selected Works of Giusseppe Peano*, University of Toronto (1973)].
Schrödinger E. (1948). *Statistical Thermodynamics*, Cambridge.
Zumino, B. (1986). Gravity theories in more than four dimensions, *Physics Reports*, **137**, 109.